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Uniqueness results for semilinear polyharmonic boundary value problems on conformally contractible domains. II

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Abstract

We continue Part I of this paper on polyharmonic boundary value problems $(-\Delta)^m u = f(u)$ on $\Omega \subset \mathbb{R}^n$, $m \in \mathbb{N}$, with Dirichlet boundary conditions. Here Ω is a bounded or unbounded conformally contractible domain as defined in Part I. The uniqueness principle proved in Part I is applied to show the following theorems: if $f(s) = \lambda s + |s|^{p-1}s$, $\lambda \leqslant 0$, with a supercritical $p > (n+2m)/(n-2m)$ we extend the well-known non-existence result of Pucci and Serrin (Indiana Univ. Math. J. 35 (1986) 681–703) for bounded star-shaped domains to the wider class of bounded conformally contractible domains. We give two examples of domains in this class which are not star-shaped. In the case where $1 < p < (n+2m)/(n-2m)$ is subcritical we give lower bounds for the L^∞-norm of non-trivial solutions. For certain unbounded conformally contractible domains, $1 < p < (n+2m)/(n-2m)$ subcritical and $\lambda \geqslant 0$ we show that the only smooth solution in $H^{2m-1}(\Omega)$ is $u \equiv 0$. Finally, on a bounded conformally contractible domain uniqueness of non-trivial solutions for $f(s) = \lambda(1 + |s|^{p-1}s)$, $p > (n+2m)/(n-2m)$, supercritical and small $\lambda > 0$ is proved. Solutions are critical points of a functional $\mathcal{L}$ on a suitable space X. The theorems are proved by finding one-parameter groups of transformations on X which strictly reduce the values of $\mathcal{L}$. Then the uniqueness principle of Part I can be applied.

Keywords: Polyharmonic operator; Uniqueness; Pohožaev's identity; Conformally contractible domains

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1. Main uniqueness results

On the smooth domain $\Omega \subset \mathbb{R}^n$ we consider for $m \in \mathbb{N}$ the boundary value problem

$$(-\Delta)^m u = f(x,u) \quad \text{in } \Omega, \qquad u = \cdots = D^{m-1}u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where for a multi-index l with $1 \leqslant |l| \leqslant m-1$ the expression $D^l u$ stands for the $n^{|l|}$-tensor consisting of the $|l|$th order derivatives of u. Only classical solutions $u \in C^{2m}(\bar{\Omega})$ of (1) are considered. They are critical points of the following functional on a suitable function space X defined later:

$$\mathcal{L}[u] = \int_\Omega \frac{1}{2}|\mathcal{D}^m u|^2 - F(x,u)\,dx,$$

where

$$\mathcal{D}^m u = \begin{cases} \Delta^r u & \text{if } m = 2r, \\ \nabla\Delta^r u & \text{if } m = 2r+1, \end{cases}$$

and $F(x,t) = \int_0^t f(x,s)\,ds$.

We describe the main uniqueness results of this paper. Let $2^* = 2n/(n-2m)$ if $n > 2m$ and $2^* = \infty$ if $n \leqslant 2m$. Recall from Definition 1 and 2 of Part I that a vector-field $\xi = (\xi_1, \ldots, \xi_m)$ in $\mathbb{R}^n$ is called *conformal* if it satisfies

$$\partial_j \xi^i + \partial_i \xi^j = \frac{2}{n}(\operatorname{div}\xi)\delta_{ij}, \quad i,j = 1,\ldots,n. \tag{2}$$

Moreover, a domain $\Omega \subset \mathbb{R}^n$ with exterior normal $\nu(x)$ on $\partial\Omega$ is called *conformally contractible* if there exists a conformal vector-field ξ such that $\xi(x)\cdot\nu(x) \leqslant 0$ on $\partial\Omega$ with strict inequality on a subset of $\partial\Omega$ of positive measure. The vector-field ξ is called an associated vector-field to Ω. For bounded conformally contractible domains the following Poincaré inequality holds.

Lemma 1 (Weighted Poincaré inequality). *Let Ω be a bounded conformally contractible domain with associated vector-field ξ such that* $\operatorname{div}\xi \leqslant 0$ *in Ω. Then there exists a value $\tilde{\lambda}_1 > 0$ such that*

$$\int_\Omega (-\operatorname{div}\xi)|\mathcal{D}^m u|^2\,dx \geqslant \tilde{\lambda}_1 \int_\Omega (-\operatorname{div}\xi)u^2\,dx \tag{3}$$

for all $u \in C_0^{m-1,1}(\bar{\Omega})$. If λ_1 denotes the first Dirichlet eigenvalue λ_1 of $(-\Delta)^m$ then the optimal value $\tilde{\lambda}$ in (3) *satisfies $\tilde{\lambda}_1 \leqslant \lambda_1$ provided $n \geqslant 3$ or $n \geqslant 2$ and $m = 1$. For domains with* $\operatorname{div}\xi = \text{const} < 0$ *one always has $\tilde{\lambda}_1 = \lambda_1$.*

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$, $n > 2m$, be a smooth bounded conformally contractible domain with associated vector-field ξ such that* $\operatorname{div}\xi \leqslant 0$ *in Ω. For $f(x,s) = \lambda s + |s|^{p-1}s$, $s \in \mathbb{R}$, problem* (1) *has only the zero solution if*

$$p \geqslant 2^* - 1, \qquad \lambda < \tilde{\lambda}_1 \frac{p(n-2m)-(n+2m)}{n(p-1)}.$$

The results also holds with strict inequality for p and equality permitted in λ. Here $\tilde{\lambda}_1$ is the weighted Poincaré constant from Lemma 1. *For $m = 1$ the result holds if equality is permitted both for p and λ.*

Remark. In Section 3 we give an extension to a class of x-dependent non-linearities $f(x,s)$.

For bounded star-shaped domains this result is due to Pohožaev [7] for $m = 1$ and Pucci and Serrin [8] for $m \geqslant 1$. For $m = 1$ the conformally contractible case was established by Reichel [9]. For $m \geqslant 2$ it is an open problem to include the case where both for p and λ equality in the above hypotheses holds, cf. [4]. For example, if $\lambda = 0$ and $p = 2^* - 1$ the only known cases are when $\Omega = B_1(0)$ and u is positive and $m \geqslant 2$ (cf. [6]) or u is radial and $m = 2, 3$ (cf. [4]).

We also mention the work of Schaaf [11], where uniqueness results are given for $m = 1$ on a different class of domains for exponents $p \geqslant p_c$, where p_c is larger than the critical Sobolev exponent. Also in the case $m = 1$ a completely different approach to uniqueness of positive solutions via maximum principles was found by Reichel and Zou [10].

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$, $n > 2m$, be a smooth bounded conformally contractible domain with associated vector-field ξ such that* $\operatorname{div}\xi \leqslant 0$ *in Ω. Let $f(x,s) = \lambda(1 + |s|^{p-1}s)$, $s \in \mathbb{R}$, with $p > 2^* - 1$ and $\lambda \geqslant 0$.*

- (i) *If $m = 1$ or $m \geqslant 2$ and $\Omega = B_1(0)$ then there exists $\bar{\lambda} > 0$ such that* (1) *has a unique* positive *solution for $\lambda \in [0, \bar{\lambda}]$.*
- (ii) *Suppose $p \geqslant 2$. If $m = 1$ and* $\operatorname{div}\xi < 0$ *in $\bar{\Omega}$ or $m \geqslant 2$ and no further restriction on* $\operatorname{div}\xi$ *then there exists $\bar{\lambda} > 0$ such that* (1) *has a unique solution for $\lambda \in [0, \bar{\lambda}]$.*

Under the same restrictions on n, m, p and λ the result holds for $f(x,s) = 1 + \lambda|s|^{p-1}s$ and $f(x,s) = \lambda e^s$. In particular (ii) *holds for $f(x,s) = \lambda e^s$ if $n > 2m$.*

Remarks. (a) Part (i) of the theorem generalizes to all those bounded domains Ω where the positivity preserving property of $(-\Delta)^m$ holds.

(b) The problem $\Delta^2 u = \lambda e^u$ in $B_1(0)$ with Dirichlet boundary conditions has recently been studied by Arioli et al. [1].

Note that the uniqueness statement of part (ii) is stronger since it does not restrict to positive solutions. However, it requires p supercritical and $p \geqslant 2$ which only holds for dimensions $n \in (2m, 6m]$. For bounded star-shaped domains and $m = 1$ the result of Theorem 3 was found by Schaaf [11]. A similar result for supercritical q-Laplacian boundary value problems on balls and $q \geqslant 2$ was found by Fleckinger and Reichel [3].

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, be a smooth bounded conformally contractible domain with associated vector-field ξ such that* $\operatorname{div}\xi \leqslant 0$ *in Ω. Let $f(x,s) = \lambda s + |s|^{p-1}s$, $s \in \mathbb{R}$, and let $\tilde{\lambda}_1$ be the weighted Poincaré constant from Lemma* 1. *Let u be a non-trivial solution of* (1).

(i) *If* $1 < p < \infty$ *and* $\lambda < \tilde\lambda_1$ *then*

$$\|u\|_\infty^{p-1} \geqslant \tilde\lambda_1 - \lambda.$$

For those domains, where $\tilde\lambda_1 = \lambda_1$ (*the first Dirichlet eigenvalue of* $(-\Delta)^m$), *the estimate shows how the solution branch bifurcating at* $\lambda = \lambda_1$ *leaves the trivial solution.*

(ii) *If* $1 < p < 2^* - 1$ *and* $\lambda < 0$ *then*

$$\|u\|_\infty^{p-1} \geqslant -\lambda \frac{2m(p+1)}{2n + (p+1)(2m-n)}.$$

In the case $n > 2m$, $\lambda < 0$, *the* L^∞*-norm of any non-trivial solution blows up as* $p \nearrow 2^* - 1$.

Finally, for unbounded domains and subcritical non-linearities, we have the following uniqueness result. This theorem is dual to Theorem 2 in the following sense: for bounded conformally contractible domains supercritical non-linearities create uniqueness, whereas for unbounded conformally contractible domains subcritical non-linearities create uniqueness.

Theorem 5. *Let* $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, *be a smooth unbounded conformally contractible domain with associated vector-field* ξ *such that* $\operatorname{div}\xi \geqslant 0$. *Let* $f(x,s) = \lambda s + |s|^{p-1}s$, $s \in \mathbb{R}$, *with* $1 < p < 2^* - 1$, $\lambda \geqslant 0$ *or* $1 < p \leqslant 2^* - 1$, $\lambda > 0$.

(i) *If* $\operatorname{div}\xi = \text{const}$ *in* Ω *then* $u \equiv 0$ *is the only smooth* $H^{2m-1}(\Omega) \cap L^{p+1}(\Omega)$*-solution of* (1).

(ii) *If only* $\operatorname{div}\xi \geqslant 0$ *is imposed then* $u \equiv 0$ *is the only smooth solution in the class*

$$\int_\Omega \bigl(1+|x|\bigr)|u|^{p+1}\,dx,\ \int_\Omega \bigl(1+|x|\bigr)|D^\gamma u|^2\,dx < \infty, \quad \forall\gamma \text{ with } |\gamma| \leqslant 2m-1.$$

Remarks. (a) For $m = 1$ and $f(x,s) = |s|^{4/(n-2)}s$ the result holds without the sign-restriction on $\operatorname{div}\xi$.

(b) In Section 6 an extension to x-dependent non-linearities $f(x,s)$ is given.

(c) In the case $m = 1$ the result of Theorem 5 was established by Reichel and Zou [10] for arbitrary positive solutions on the complement of bounded star-shaped domains in $\mathbb{R}^n$ without any integrability assumption.

Comments on the case $n = 2$. The dimension $n = 2$ is only of interest in Theorems 4 and 5. It was shown in Part I that all simply connected domains in $n = 2$ are conformally contractible. Since it is well known that the Laplacian operator commutes with conformal maps in dimension $n = 2$ it is therefore not surprising that for $m = 1$ both theorems continue to hold, if in part (ii) of Theorem 5 the weight $(1+|x|)$ is replaced by $(1+\operatorname{div}\xi)$. For $m \geqslant 2$ however, both theorems only hold for the restricted class of conformally contractible domains, where $\operatorname{div}\xi$ is a linear function, i.e., ξ is of the type described in part (b) of Lemma 7, Part I. No changes in the theorems are necessary.

All four theorems have one common source: the uniqueness principle of Part I of this paper. If a one-parameter transformation group $g_\epsilon : X \to X$ acts on $\mathcal{L}$ by strictly reducing its values then the transformation group is called a *variational subsymmetry*. As shown in Part I variational subsymmetries ensure uniqueness of the critical point of $\mathcal{L}$, cf. Section 2 for more details.

Function spaces. We use the same function spaces as in Part I. Let $C_0^m(\bar{\Omega})$ be the subspace of $C^m(\bar{\Omega})$ consisting of those functions satisfying Dirichlet boundary conditions of order $m-1$, i.e., $u = \cdots = D^{m-1}u = 0$ on $\partial\Omega$. By $H^k(\Omega)$, $k \in \mathbb{N}$, we denote the space of L^2-functions having L^2-integrable derivatives of order up to k. The norm in $H^k(\Omega)$ is

$$\|u\|_{H^k} = \sum_{|\gamma|\leqslant k} \left(\int_\Omega |D^\gamma u|^2 \, dx \right)^{1/2}.$$

If an additional positive weight function ω is introduced then the weighted spaces are denoted by $H^k(\Omega;\omega)$.

If Ω is bounded then a suitable function space, on which the functional $\mathcal{L}$ is still well defined, is the space $X = C_0^{m-1,1}(\bar{\Omega})$ of functions with continuous derivatives of order up to $m-1$, such that the derivatives of order $m-1$ are Lipschitz continuous, and the Dirichlet condition of order $m-1$ on $\partial\Omega$ holds. In the cases considered in Theorem 5 where Ω is unbounded, the functional is well defined on the space $X = C_0^{m-1,1}(\bar{\Omega}) \cap H^m(\Omega) \cap L^{p+1}(\Omega)$.

Notation. We use standard multi-index notation $\gamma = (\gamma_1, \dots, \gamma_n)$, $\gamma_i \in \mathbb{N}_0$, with $|\gamma| = \gamma_1 + \cdots + \gamma_n$ and

$$D^\gamma = \frac{\partial^{|\gamma|}}{\partial_{x_1}^{\gamma_1} \dots \partial_{x_n}^{\gamma_n}}$$

with ∂_i a shorthand for ∂/∂_{x_i}. Note that D^γ only acts on the function immediately following the symbol, i.e., $D^\gamma u D^\delta v$ stands for $(D^\gamma u)(D^\delta v)$. This also holds, e.g., for $\nabla\Delta^r\partial_i u \cdot \nabla\Delta^s v = (\nabla\Delta^r\partial_i u)\cdot(\nabla\Delta^s v)$. For a vector function ξ we write $D\xi$ for the Jacobian matrix $(\partial_j\xi_i)_{ij}$, and for a scalar function ϕ we write $D^2\phi$ for its Hessian matrix $(\partial_{ij}^2\phi)_{ij}$.

Organization of the paper is as follows. In Section 2 we recall the main uniqueness principle shown in Part I of this paper. The following 4 sections are devoted to the proof of the main results of Theorems 2–5. Finally, in Section 7 we prove the Poincaré inequality of Lemma 1.

2. The uniqueness principle of Part I

In Part I we have studied one-parameter family of maps

$$G = \left\{ g_\epsilon : C_0^{m-1,1}(\bar{\Omega}) \to C_0^{m-1,1}(\bar{\Omega}) \right\}_{\epsilon\in\mathbb{R}}$$

with the group-property $g_{\epsilon_1} \circ g_{\epsilon_2} = g_{\epsilon_1+\epsilon_2}$, $g_0 = \mathrm{Id}$. They arise from the ODE system

$$\dot{X} = \xi(X), \qquad X(0) = x, \tag{4}$$

$$\dot{U} = \alpha U \operatorname{div}\xi(X), \qquad U(0) = u, \tag{5}$$

where $\alpha \in \mathbb{R}$ and ξ is a conformal vector-field in $\mathbb{R}^n$, $n \geqslant 3$. We denote by $g_\epsilon(x,u) = (\chi_\epsilon(x), \psi_\epsilon(x,u))$ the solution of (4)–(5) at time ϵ. Due to the explicit form of the conformal vector-fields, cf. Lemma 7 in Part I, we can integrate (4) explicitly. It is then easy to show that the solutions of (4)–(5) exist for all $\epsilon \in \mathbb{R}$, and that if $u \in C_0^m(\bar{\Omega})$ then $g_\epsilon(\Gamma_u)$ represents the graph of a new function $g_\epsilon u \in C_0^m(g_\epsilon \bar{\Omega})$.

This implies the following formula for the transformed function $\tilde{u}(\tilde{x})$:

$$\tilde{u}(\tilde{x}) = \psi_\epsilon(\mathrm{Id} \times u)\big[\chi_\epsilon(\mathrm{Id})\big]^{-1}(\tilde{x}), \quad \tilde{x} \in \tilde{\Omega}, \tag{6}$$

which defines the map

$$g_\epsilon : u \mapsto \tilde{u}$$

for functions $u \in C_0^{m-1,1}(\bar{\Omega})$ and for ϵ in a small interval containing 0. The transformed function $\tilde{u}$ is defined on the transformed domain $\tilde{\Omega} = \chi_\epsilon(\Omega)$. For the transformed function we use the notation $g_\epsilon u$ as well as $\tilde{u}(\tilde{x})$, and for its domain of definition we write $g_\epsilon \Omega$ as well as $\tilde{\Omega}$. The essence of Sections 2–3 in Part I is as follows.

Proposition 6. *Suppose Ω is a smooth conformally contractible domain and let $u \in C_0^m(\bar{\Omega})$ or $u \in C_0^{m-1,1}(\bar{\Omega})$. For $\epsilon > 0$ the transformed function $g_\epsilon u$ is defined on $g_\epsilon \Omega \subset \Omega$. If we extend $g_\epsilon u$ by 0 outside $g_\epsilon \Omega$ then $g_\epsilon u \in C_0^{m-1,1}(\bar{\Omega})$.*

Remark. Note that although starting in $C_0^m(\bar{\Omega})$ we only end in $C_0^{m-1,1}(\bar{\Omega})$ since the Dirichlet conditions of order $m-1$ and the extension by 0 only take care of the C^{m-1}-smoothness of the transformed function.

We recall the main uniqueness result of Theorem 4 in Part I.

Definition 7. Let Ω be a conformally contractible domain with associated vector field ξ and suppose the functional $\mathcal{L} : X = C_0^{m-1,1}(\bar{\Omega}) \cap H^m(\Omega) \cap L^{p+1}(\Omega) \to \mathbb{R}$ is well defined. Let $G = \{g_\epsilon\}_{\epsilon \geqslant 0}$ be the transformation group generated by ξ. The group G is called a strict variational subsymmetry for the functional $\mathcal{L}$ if there exists a point $u_0 \in X$ such that

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\bigg|_{\epsilon=0} < 0 \quad \text{for all } u \in X \setminus \{u_0\}.$$

Theorem 8 (Uniqueness principle). *Let Ω be a conformally contractible domain with associated vector-field ξ and let $\mathcal{L} : X \to \mathbb{R}$ be as in Section 1. Suppose that the transformation group $G = \{g_\epsilon\}_{\epsilon \geqslant 0}$ generated by ξ is a strict variational subsymmetry with respect to u_0. Then u_0 is the only possible critical point of $\mathcal{L}$ within the following class of functions*:

(i) *u smooth in case Ω is bounded,*

(ii) *$F(x,u)\operatorname{div}\xi$, $f(x,u)u\operatorname{div}\xi$, $\xi\cdot\nabla_x F(x,u)\in L^1(\Omega)$ and $u\in H^{2m-1}(\Omega;\omega)$ if Ω is unbounded, where the weight $\omega=1$ if $\operatorname{div}\xi=\text{const}$ and $\omega=(1+|x|)$ if no restriction on $\operatorname{div}\xi$ is imposed.*

For the application of Theorem 8 to a specific functional one has to verify a strict variational subsymmetry. In all of our applications this is done through the following rate of change formula.

Theorem 9 (Rate of change formula). *Suppose Ω is a conformally contractible domain with associated vector field ξ. Let $G=\{g_\epsilon\}_{\epsilon\geqslant 0}$ be the transformation group generated by ξ. Let $u\in C_0^m(\bar{\Omega})\cap H^m(\Omega;\omega)$, where the weight $\omega=1$ if $\operatorname{div}\xi=\text{const}$ and $\omega=(1+|x|)$ if no restriction on $\operatorname{div}\xi$ is imposed. Then the rate of change of the functional $\mathcal{L}$ under the transformation group can be computed as follows*:

$$\begin{aligned}\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\bigg|_{\epsilon=0}&=\int_\Omega\left(\alpha-\frac{m}{n}+\frac{1}{2}\right)(\operatorname{div}\xi)|\mathcal{D}^m u|^2\,dx\\&\quad-\int_\Omega\big(\alpha f(x,u)u+F(x,u)\big)\operatorname{div}\xi+\xi\cdot\nabla_x F(x,u)\,dx,\end{aligned}\tag{7}$$

provided the last volume integral exists. This is, e.g., the case for non-linearities $|f(x,s)|\leqslant C(1+|s|^p)$ provided $u\in L^{p+1}(\Omega;\omega)$.

3. Proof of Theorem 2

For the proof of Theorem 2 we have $f(x,s)=\lambda s+|s|^{p-1}s$. We need to verify that (4)–(5) generate a strict variational subsymmetry w.r.t. 0. We choose $\alpha=-1/(p+1)$ and find from the rate of change formula (7) of Theorem 9 that

$$\begin{aligned}\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\bigg|_{\epsilon=0}&=\int_\Omega\left(\frac{1}{p+1}+\frac{m}{n}-\frac{1}{2}\right)(-\operatorname{div}\xi)|\mathcal{D}^m u|^2\,dx\\&\quad+\int_\Omega\lambda\left(\frac{1}{2}-\frac{1}{p+1}\right)u^2(-\operatorname{div}\xi)\,dx.\end{aligned}$$

By the super-criticality assumption the coefficient of $|\mathcal{D}^m u|^2(-\operatorname{div}\xi)$ is non-positive. Hence, with the weighted Poincaré inequality from Lemma 1 we get

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\bigg|_{\epsilon=0}\leqslant\int_\Omega\left\{\tilde{\lambda}_1\left(\frac{1}{p+1}+\frac{m}{n}-\frac{1}{2}\right)+\lambda\left(\frac{1}{2}-\frac{1}{p+1}\right)\right\}u^2(-\operatorname{div}\xi)\,dx,$$

where $\tilde{\lambda}_1$ is the Poincaré constant. The assumption of Theorem 2 implies that the coefficient in $\{\ldots\}$ is strictly negative. Hence the group $G=\{g_\epsilon\}_{\epsilon\geqslant 0}$ is a strict variational subsymmetry w.r.t. $u_0=0$. Theorem 8 shows that $u\equiv 0$ is the only solution of (1). $\square$

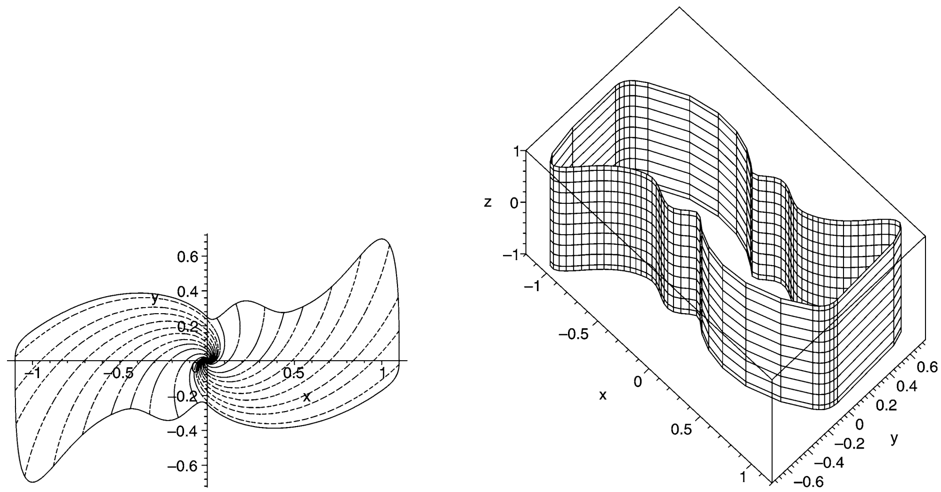

Fig. 1. 2d-cut and 3d-domain.

Remark. There is a fundamental difficulty in extending the result of Theorem 2 to the equality case $\lambda = 0$ and $p = (n + 2m)/(n - 2m)$. In this case, by Theorem 14, Part I (Pohožaev's identity), we obtain $\mathcal{D}^m u = 0$ on a subset of $\partial\Omega$ of positive measure. For $m = 1$ the unique continuation theorem, cf. [5], shows $u \equiv 0$. For $m > 1$ the unique continuation theorem requires the vanishing of all derivatives up to order $2m - 1$. Except for the cases mentioned in Section 1 this gap has not been closed.

We give two examples of domains in $\mathbb{R}^n$, $n \geqslant 3$, which are not star-shaped but conformally contractible, and the associated vector-field ξ satisfies $\operatorname{div}\xi \leqslant 0$.

Example 1. The vector-field $\xi = (-x + y, -y - x, -z)$ with $\operatorname{div}\xi = -3$ generates a composition of a dilation and a rotation in the (x, y)-plane. We construct a conformally contractible domain by extending the 2d-domain Ω_2 cylindrically in the z-direction to a 3d-domain Ω_3, cf. Fig. 1. In Ω_2 the trajectories of $(\dot{x}, \dot{y}) = (-x + y, -y - x)$ starting from the boundary are shown. Notice that Ω_2 is positively-invariant under the flow, i.e., Ω_2 is conformally contractible in the plane. By cylindrical extension this remains true for Ω_3.

In n dimensions we extend Ω_2 to $\Omega_n = \Omega_2 \times B_1^{n-2}(0)$, where $B_1^{n-2}(0)$ is an $(n - 2)$-dimensional ball of radius 1. The associated vector-field is $\xi = (-x_1 + x_2, -x_2 - x_1, -x_3, \ldots, -x_n)$ with $\operatorname{div}\xi = -n$.

Example 2. The vector-field $\xi = (-2xz, -2yz, -z^2 + x^2 + y^2)$ with $\operatorname{div}\xi = -6z$ generates a one-parameter group of conformal maps involving inversions. We construct a conformally contractible domain by rotating a planar domain around the z-axis. The flow $(\dot{x}, \dot{y}, \dot{z}) = \xi(x, y, z)$ is rotation-symmetric around the z-axis. Figure 2 shows the 2d-cut and the trajectories starting from the boundary. The 2d-domain Ω_2 is positively-invariant, and by rotation-symmetry also the 3d-domain Ω_3. Hence Ω_3 is conformally contractible,

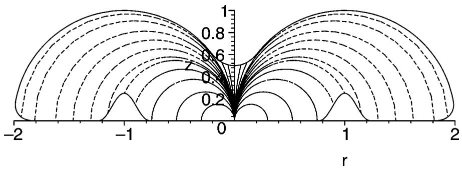

Fig. 2. 2d-domain and trajectories.

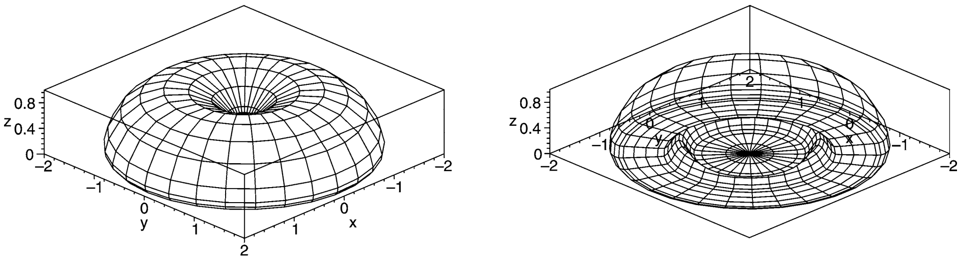

Fig. 3. 3d-domain from above and below.

and since it lies in the region $z \geqslant 0$ we have $\operatorname{div}\xi \leqslant 0$. Figure 3 shows Ω_3 from above and below.

In n dimensions we rotate the boundary curve around the x_n-axis; the (almost flat) basis will be in the $(x_1, \ldots, x_{n-1})$-hyperplane. The associated vector-field is $\xi = (-2x_1x_n, \ldots, -2x_{n-1}x_n, -x_n^2 + x_1^2 + \cdots + x_{n-1}^2)$ with $\operatorname{div}\xi = -2nx_n$.

Theorem 2′. *Let $\Omega \subset \mathbb{R}^n$, $n > 2m$, be a smooth bounded conformally contractible domain with associated vector-field ξ such that $\operatorname{div}\xi \leqslant 0$ in Ω. Let $(\chi_\epsilon(x), \psi_\epsilon(x,s))$ be the solution to* (4)–(5). *Suppose*

$$\frac{F(\chi_\epsilon(x), \psi_\epsilon(x,s))}{|\psi_\epsilon(x,s)|^{2n/(n-2m)}} \text{ strictly increasing in } \epsilon \text{ for almost all } x, s \in \bar{\Omega} \times \mathbb{R}. \tag{8}$$

Then the only smooth solution of (1) *is $u \equiv 0$. Examples for* (8) *are*

(a) $f(x,s) = |\xi(x)|^\beta |s|^{\gamma-1} s$ *with* $\beta \geqslant 0$ *and* $\gamma > (n + 2m + 2\beta)/(n - 2m)$,
(b) $f(s)$ *such that* $F(s)/|s|^{2n/(n-2m)}$ *strictly increasing in* s.

The proof follows the lines of the proof of Theorem 2. We choose $\alpha = m/n - 1/2$. Then

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\bigg|_{\epsilon=0} = -\int\limits_\Omega \left(\frac{2m-2}{2n} f(x,u)u + F(x,u)\right) \operatorname{div}\xi + \xi \cdot \nabla_x F(x,u)\, dx.$$

Differentiating (8) w.r.t. ϵ at $\epsilon = 0$ we obtain

$$\xi \cdot \nabla_x F(x,s) + \operatorname{div}\xi \left(F(x,s) + \frac{2m-n}{2n} f(x,s)s\right) > 0$$

for almost all $(x,s) \in \bar{\Omega} \times \mathbb{R}$. This shows that we have a strict variational subsymmetry with respect to 0.

4. Proof of Theorem 3

We have $f(x,s) = \lambda(1 + s^{(p)})$, $s \in \mathbb{R}$, where for simplicity we use the notation $t^{(p)} = |t|^{p-1}t$ for the odd pth power. For $\lambda = 0$ problem (1) has the unique solution $u \equiv 0$. For small $\lambda \in [0, \tilde{\lambda}]$ let u_λ be the solution of (1) obtained from the implicit function theorem.

Lemma 10. (a) *If $m = 1$ or if $m \geqslant 2$ and $\Omega = B_1(0)$ then u_λ is the minimal positive solution.*

(b) *If U is the solution of $(-\Delta)^m U = 1$ in Ω with Dirichlet boundary conditions, then $\|u_\lambda - \lambda U\|_{C^{2m+\alpha}(\bar{\Omega})} = O(\lambda^{p+1})$ as $\lambda \to 0$.*

Proof. (a) Under the conditions of Theorem 3 the operator $(-\Delta)^m$ is positivity preserving, i.e., if $(-\Delta^m)w = f$ in Ω with Dirichlet boundary conditions on $\partial\Omega$ and $f \geqslant 0$, $f \not\equiv 0$, then $w > 0$ in Ω, cf. [2] for $m \geqslant 2$ on balls. By the implicit function theorem $\|u_\lambda\|_{C^{2m+\alpha}(\bar{\Omega})} \to 0$ as $\lambda \to 0$. So for small $\lambda > 0$ we have $(-\Delta)^m u_\lambda \geqslant 0$ in Ω and hence $u_\lambda > 0$ in Ω. Now fix $\lambda > 0$. We can start the monotone iteration scheme with the subsolution $u_0 \equiv 0$ and define $(-\Delta^m)u_{k+1} = \lambda(1 + u_k^p)$ in Ω with Dirichlet boundary conditions. We obtain the sequence $0 < u_k < u_{k+1} < u_\lambda$, and by monotonicity we get the minimal positive solution $\underline{u}_\lambda = \lim_{k\to\infty} \leqslant u_\lambda$. Since $\underline{u}_\lambda \to 0$ as $\lambda \to 0$ the uniqueness part of the implicit function theorem implies $\underline{u}_\lambda = u_\lambda$ for $\lambda > 0$ small.

(b) Since u_λ is increasing in λ we have that $\|u_\lambda\|_\infty$ stays bounded as $\lambda \to 0$. Let M be so large that $(1 + \|u_\lambda\|_\infty^p) \leqslant M$ for $\lambda \in [0, \tilde{\lambda}]$. Then $\lambda M U$ is a supersolution to (1), i.e., $u_\lambda \leqslant \lambda M U$, and hence $\|u_\lambda\|_\infty = O(\lambda)$ as $\lambda \to 0$. If we define $w_\lambda = u_\lambda/\lambda$ then w_λ is uniformly bounded and satisfies $(-\Delta^m)w_\lambda = 1 + \lambda^p w_\lambda^p$ in Ω with Dirichlet boundary conditions. Therefore $w_\lambda \to U$ in $C^{2m+\alpha}(\bar{\Omega})$ as $\lambda \to 0$. This finishes part (b). $\square$

Now we can start the proof of Theorem 3. If u is also a solution of (1) then let $v = u - u_\lambda$. The corresponding boundary value problem for v is given by

$$(-\Delta)^m v + \lambda(v + u_\lambda)^{(p)} - \lambda u_\lambda^{(p)} = 0 \quad \text{in } \Omega, \qquad v = \cdots = D^{m-1} v = 0 \quad \text{on } \partial\Omega.$$

The corresponding functional is given by

$$\mathcal{L}[v] = \int_\Omega \frac{1}{2} |\mathcal{D}^m v|^2 - G(x,v)\,dx,$$

where

$$G(x,s) = \frac{\lambda}{p+1} \left| s + u_\lambda(x) \right|^{p+1} - \lambda u_\lambda(x)^{(p)} s - \frac{\lambda}{p+1} \left| u_\lambda(x) \right|^{p+1}.$$

The last term has been added in order to have $G(x,0)=0$. Let ξ be an associated conformal vector-field such that $\xi\cdot\nu\leqslant 0$ on $\partial\Omega$ and $\operatorname{div}\xi\leqslant 0$ in $\bar\Omega$. We fix a negative value α such that

$$\frac{1}{p+1}<-\alpha<\frac{n-2m}{2n}.$$

We need to verify that ξ generates a strict variational subsymmetry w.r.t. 0. By the rate of change formula from Theorem 9 we obtain

$$\begin{aligned}\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon v]\bigg|_{\epsilon=0}=&\int_\Omega\left(\alpha-\frac{m}{n}+\frac{1}{2}\right)(\operatorname{div}\xi)|\mathcal{D}^m v|^2\,dx\\&+\int_\Omega\lambda\xi\cdot\nabla u_\lambda\big(-(v+u_\lambda)^{(p)}+p|u_\lambda|^{p-1}v+u_\lambda^{(p)}\big)\,dx\\&+\int_\Omega\lambda\alpha(-\operatorname{div}\xi)\big((v+u_\lambda)^{(p)}-(u_\lambda)^{(p)}\big)v\,dx\\&+\int_\Omega\frac{\lambda}{p+1}(-\operatorname{div}\xi)\big(|v+u_\lambda|^{p+1}-(p+1)u_\lambda^{(p)}v-|u_\lambda|^{p+1}\big).\end{aligned}\tag{9}$$

Define the functions $h_1,h_2:[0,\tilde\lambda]\times\Omega\times\mathbb{R}\to\mathbb{R}$ by

$$\begin{aligned}h_1(\lambda,x,s)&=\xi\cdot\nabla u_\lambda\big(-(s+u_\lambda)^{(p)}+p|u_\lambda|^{p-1}s+u_\lambda^{(p)}\big),\\h_2(\lambda,x,s)&=\alpha(-\operatorname{div}\xi)\big((s+u_\lambda)^{(p)}-u_\lambda^{(p)}\big)s\\&\quad-\frac{\operatorname{div}\xi}{p+1}\big(|s+u_\lambda|^{p+1}-(p+1)u_\lambda^{(p)}s-|u_\lambda|^{p+1}\big),\end{aligned}$$

and let $h(\lambda,x,s)=h_1(\lambda,x,s)+h_2(\lambda,x,s)$. We discuss the behaviour of h_1,h_2 depending on the different types of hypotheses. Recall that $\operatorname{div}\xi\leqslant 0$ is a linear function in Ω, i.e., it has at most first order zeroes at $\partial\Omega$.

Case (i). We restrict attention to positive solutions of (1). Since u_λ is the minimal positive solution we can assume $v\geqslant 0$, i.e., we discuss h_1,h_2 for $s\geqslant 0$. We split the domain $\bar\Omega=D_1\cup D_2$, where D_1 is a compact subset of Ω and D_2 a neighbourhood of $\partial\Omega$ such that $\xi\cdot\nabla u_\lambda\geqslant 0$. For positive s convexity implies $(-(s+u_\lambda)^{(p)}+p|u_\lambda|^{p-1}s+u_\lambda^{(p)})\leqslant 0$. Hence,

$$h_1(\lambda,x,s)\leqslant 0\quad\text{for }(\lambda,x,s)\in[0,\tilde\lambda]\times[0,\infty)\times D_2.$$

On the set D_1 we have $|\xi\cdot\nabla u_\lambda|\leqslant c\lambda(-\operatorname{div}\xi)$ for a suitable constant $c>0$. Hence we have

$$h_1(\lambda,x,s)\leqslant C(1+s^p)(-\operatorname{div}\xi)\quad\text{in }D_1$$

for all $s\geqslant 0$ uniformly for $\lambda\in[0,\tilde\lambda]$. Next we use the mean value theorem to get $-(s+u_\lambda)^{(p)}+p|u_\lambda|^{p-1}s+u_\lambda^{(p)}=-p(p-1)|\lambda\theta|^{p-2}s^2$ for some $\theta\in[u_\lambda(x),u_\lambda(x)+s]$.

Since $u_\lambda = \lambda U + O(\lambda^{p+1})$ by Lemma 10 and since U is positive in D_1 we find that $|h_1(\lambda, x, s)| \leqslant c|\operatorname{div}\xi|\lambda^{p-1}s^2$ for small $s > 0$, i.e.,

$$h_1(\lambda, x, s)\big/\big(|\operatorname{div}\xi|s^2\big) \text{ stays bounded as } s \to 0$$

uniformly for $(\lambda, x) \in [0, \tilde{\lambda}] \times D_1$.

The discussion of h_2 is simpler. Since $\alpha < -1/(p+1)$ and $(-\operatorname{div}\xi) \geqslant 0$ we have that $h_2(\lambda, x, s) \leqslant (C - |s|^{p+1})(-\operatorname{div}\xi)$ uniformly for $(\lambda, x) \in [0, \tilde{\lambda}] \times \bar{\Omega}$. Likewise $h_2(\lambda, x, s)/(|\operatorname{div}\xi|s^2)$ is bounded as $s \to 0$ uniformly for $(\lambda, x) \in [0, \tilde{\lambda}] \times \bar{\Omega}$. Since for large positive s the negative function $h_2(\lambda, x, s)$ dominates $h_1(\lambda, x, s)$ we have for their sum

$$h(\lambda, x, s) \leqslant As^2(-\operatorname{div}\xi) \tag{10}$$

for all $(\lambda, x, s) \in [0, \tilde{\lambda}] \times [0, \infty) \times \bar{\Omega}$ with a suitable constant A.

Case (ii). Now we allow also sign-changing solution of (1), i.e., we have to consider h_1, h_2 for $s \in \mathbb{R}$. Since $\operatorname{div}\xi$ has first order zeroes at most at $\partial\Omega$, we can estimate $|\xi \cdot \nabla u_\lambda| \leqslant c(-\operatorname{div}\xi)$ for a suitable constant $c > 0$. This is possible for $m \geqslant 2$ due to the Dirichlet boundary conditions and for $m = 1$ under the extra requirement $\operatorname{div}\xi < 0$ in $\bar{\Omega}$. For $s \to \pm\infty$ we find that $h_1(\lambda, x, s) \leqslant C(1 + |s|^p)(-\operatorname{div}\xi)$. If $p \geqslant 2$ then Taylor's theorem shows that $h_1(\lambda, x, s)/(|\operatorname{div}\xi|s^2)$ stays bounded as $s \to 0$ uniformly for $(\lambda, x) \in [0, \tilde{\lambda}] \times \Omega$. The discussion of h_2 is exactly the same as in Case (i). Thus we reach the same conclusion (10) as in Case (i) for all $(\lambda, x, s) \in [0, \tilde{\lambda}] \times \mathbb{R} \times \bar{\Omega}$.

In both cases we may estimate (9) by

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon v]\bigg|_{\epsilon=0} \leqslant \int\limits_\Omega \left(\alpha - \frac{m}{n} + \frac{1}{2}\right)(\operatorname{div}\xi)|\mathcal{D}^m v|^2 + \lambda A v^2(-\operatorname{div}\xi)\,dx.$$

Since $-\alpha < (n-2m)/(2n)$ we continue by the weighted Poincaré inequality from Lemma 1 and get

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon v]\bigg|_{\epsilon=0} \leqslant \int\limits_\Omega \left(\alpha - \frac{m}{n} + \frac{1}{2}\right)\tilde{\lambda}_1 v^2 \operatorname{div}\xi + \lambda A v^2(-\operatorname{div}\xi)\,dx,$$

where $\tilde{\lambda}_1$ is the Poincaré constant. This shows that for $\lambda > 0$ sufficiently small we have a strict variational subsymmetry w.r.t. 0. By Theorem 8 $v \equiv 0$, i.e., $u \equiv u_\lambda$ is the only critical point of $\mathcal{L}$ for sufficiently small λ. $\square$

5. Proof of Theorem 4

We will show that every solution u of (1) with small L^∞-norm is trivial. From the rate of change formula (7) in Theorem 9 we obtain

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\bigg|_{\epsilon=0} = \int\limits_\Omega \left(\alpha - \frac{m}{n} + \frac{1}{2}\right)(\operatorname{div}\xi)|\mathcal{D}^m u|^2\,dx + \int\limits_\Omega (-\operatorname{div}\xi)\left(\alpha + \frac{1}{p+1}\right)|u|^{p+1} + (-\operatorname{div}\xi)\left(\alpha + \frac{1}{2}\right)\lambda u^2\,dx.$$

We choose

$$-\alpha \leqslant \min\left\{\frac{1}{p+1}, \frac{n-2m}{2n}\right\}.$$

Thus the coefficient of $(-\operatorname{div}\xi)|\mathcal{D}^m u|^2$ is non-positive and the coefficient of $(-\operatorname{div}\xi) \times |u|^{p+1}$ is non-negative. Hence we can apply the weighted Poincaré inequality from Lemma 1 and get

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\bigg|_{\epsilon=0} \leqslant \int_\Omega \left[\tilde{\lambda}_1\left(-\alpha + \frac{m}{n} - \frac{1}{2}\right) + \left(\alpha + \frac{1}{2}\right)\lambda\right](-\operatorname{div}\xi)u^2\,dx + \int_\Omega \left(\alpha + \frac{1}{p+1}\right)\|u\|_\infty^{p-1}(-\operatorname{div}\xi)u^2\,dx,$$

where $\tilde{\lambda}_1$ is the Poincaré constant. Thus we have a strict variational subsymmetry w.r.t. 0 provided

$$\|u\|_\infty^{p-1} < \frac{\tilde{\lambda}_1(\alpha + (n-2m)/(2n)) - \lambda(\alpha + 1/2)}{\alpha + 1/(p+1)}, \tag{11}$$

i.e., any solution satisfying (11) is trivial by the uniqueness principle of Theorem 8. In turn, any non-trivial solution has to satisfy the reverse inequality in (11). If we let $\alpha \to -\infty$ then we obtain part (i) of the theorem. Part (ii) follows if we take $\alpha = (2m-n)/(2n)$. □

6. Proof of Theorem 5

For Theorem 5 we have $f(x,s) = \lambda s + |s|^{p-1}s$. Hence $\mathcal{L}$ is well defined on $X = C_0^{m-1,1}(\bar{\Omega}) \cap H^m(\Omega) \cap L^{p+1}(\Omega)$. To show a strict variational subsymmetry, we can follow the proof of Theorem 2. By choosing $\alpha = -1/(p+1)$ in (7) we get the rate of change formula

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\bigg|_{\epsilon=0} = \int_\Omega \left(\frac{1}{p+1} + \frac{m}{n} - \frac{1}{2}\right)(-\operatorname{div}\xi)|\mathcal{D}^m u|^2 + \lambda\left(\frac{1}{2} - \frac{1}{p+1}\right)u^2(-\operatorname{div}\xi)\,dx.$$

By assumption $\operatorname{div}\xi \geqslant 0$. Since p is subcritical and $\lambda \geqslant 0$ (with one strict inequality) it is easy to see that we have a strict variational subsymmetry with respect to 0. The uniqueness principle of Theorem 8 completes the claim. □

For x-dependent non-linearities we have the following generalization of Theorem 5.

Theorem 5′. *Let $\Omega \subset \mathbb{R}^n$ be a smooth unbounded conformally contractible domain with associated vector-field ξ such that* $\operatorname{div}\xi \geqslant 0$. *Let $(\chi_\epsilon(x), \psi_\epsilon(x,s))$ be the solution to* (4)–(5) . *Suppose*

$$\frac{F(\chi_\epsilon(x), \psi_\epsilon(x,s))}{|\psi_\epsilon(x,s)|^{2n/(n-2m)}} \text{ strictly decreasing in } \epsilon \text{ for almost all } x, s \in \bar{\Omega} \times \mathbb{R}.$$

Moreover let $\omega = 1$ if $\operatorname{div}\xi = \text{const}$ and $\omega = (1+|x|)$ if no restriction on $\operatorname{div}\xi$ is imposed. Then the only smooth solution u of (1) *satisfying $u \in H^{2m-1}(\Omega;\omega)$ and $F(x,u)\operatorname{div}\xi$, $f(x,u)u\operatorname{div}\xi$, $\xi\cdot\nabla_x F(x,u) \in L^1(\Omega)$ is $u \equiv 0$.*

Remark. The complement of the domain in Example 1 is unbounded and conformally contractible, where the associated vector-field satisfies $\operatorname{div}\xi = 3$. Hence the above results apply. The complement of the domain in Example 2 is also unbounded and conformally contractible, but the associated vector-field ξ satisfies $\operatorname{div}\xi = 6z$, which is sign-changing. Hence neither Theorem 5 nor Theorem 5′ applies. However, the half-space $x_n > 0$ with $\xi = (2x_1x_n, \dots, 2x_{n-1}x_n, x_n^2 - x_1^2 - \cdots - x_{n-1}^2)$ provides non-trivial examples for Theorem 5′ if we take $f(x,s) = |\xi(x)|^\beta |s|^{\gamma-1}s$ with $\beta \geqslant 0$ and $1 < \gamma < (n+2m+2\beta)/(n-2m)$.

7. Proof of the weighted Poincaré inequality

For $n \geqslant 3$ the function $-\operatorname{div}\xi$ is linear and non-negative. Hence we may suppose that after a rotation of the coordinate system we have $-\operatorname{div}\xi = a + bx_1 \geqslant 0$ in Ω. To avoid trivialities assume $b < 0$ and $x_1 \leqslant -a/b$ for $x \in \Omega$ (a similar proof holds if $b < 0$ and $x_1 \geqslant -a/b$). Let C denote a generic constant. First, we find

$$\int_\Omega (a+bx_1)u^2\,dx \leqslant C\int_\Omega u^2\,dx = \frac{-C}{b}\int_\Omega (a+bx_1)\partial_{x_1}u^2\,dx$$
$$\leqslant C\left(\int_\Omega (a+bx_1)u^2\,dx\right)^{1/2}\left(\int_\Omega (a+bx_1)|\partial_{x_1}u|^2\,dx\right)^{1/2},$$

i.e., $\int_\Omega (a+bx_1)u^2\,dx \leqslant C\int_\Omega (a+bx_1)|\partial_{x_1}u|^2\,dx$. Likewise, for $i = 2,\dots,n$ we find

$$\int_\Omega (a+bx_1)u^2\,dx = -\int_\Omega x_i\,\partial_{x_i}\big((a+bx_1)u^2\big)\,dx \leqslant C\int_\Omega (a+bx_1)u\,\partial_{x_i}u\,dx$$
$$\leqslant C\left(\int_\Omega (a+bx_1)u^2\,dx\right)^{1/2}\left(\int_\Omega (a+bx_1)|\partial_{x_i}u|^2\,dx\right)^{1/2}.$$

Hence $\int_\Omega (a+bx_1)u^2\,dx \leqslant C\int_\Omega (a+bx_1)|\partial_{x_i}u|^2\,dx$ for $i = 2,\dots,n$. The result in (3) of Lemma 1 now follows from iterating these inequalities. To find the relation of the best constant $\tilde\lambda$ in (3) and λ_1, let ϕ_1 be the first Dirichlet eigenfunction and suppose m is even. Then, by formula (10) in the proof of Lemma 9, Part I,

$$\int_\Omega (-\operatorname{div}\xi)\Delta^r\phi_1\Delta^r\phi_1\,dx$$
$$= \int_\Omega \Delta^r\big((-\operatorname{div}\xi)\phi_1\big)\Delta^r\phi_1\,dx - \int_\Omega 2r\beta_i\,\Delta^{r-1}\partial_i\phi_1\Delta^r\phi_1\,dx.$$

The second integral vanishes, as can be seen directly or explicitly through Lemma 11, Part I. Hence integration by parts yields

$$\int_{\Omega} (-\operatorname{div}\xi)\Delta^r\phi_1\Delta^r\phi_1\,dx = \lambda_1 \int_{\Omega} (-\operatorname{div}\xi)\phi_1^2\,dx,$$

which shows that the optimal constant $\tilde{\lambda}_1$ is smaller or equal to λ_1. The proof for m odd is similar. Finally, in the case $n = 2$ the function $-\operatorname{div}\xi \geqslant 0$ is harmonic. Hence it also has at most simple zeroes on $\partial\Omega$ and we can estimate it from above and below by a linear function. However, for $n = 2$ and $m \geqslant 2$ the relation between the optimal constant $\tilde{\lambda}_1$ and λ_1 is not clear. For $n = 2$ and $m = 1$ a similar proof as above works and shows $\tilde{\lambda}_1 \geqslant \lambda_1$. $\square$

References

[1] G. Arioli, F. Gazzola, H.-Ch. Grunau, E. Mitidieri, A semilinear fourth order elliptic problem with exponential nonlinearity, Preprint No. 35, Fakultät für Mathematik, Universität Magdeburg, 2002.

[2] T. Boggio, Sulle funzioni di Green d'ordine m, Rend. Circ. Mat. Palermo 20 (1905) 97–135.

[3] J. Fleckinger, W. Reichel, Global solution branches for p-Laplacian boundary value problems, Preprint, 2003.

[4] H.-Ch. Grunau, Polyharmonische Dirichletprobleme: Positivität, kritische Exponenten und kritische Dimensionen, Habilitationsschrift, Universität Bayreuth, 1996.

[5] C. Miranda, Partial Differential Equations of Elliptic Type, 2nd ed., Springer-Verlag, 1970.

[6] P. Oswald, On a priori estimates for positive solutions of a semilinear biharmonic equation in a ball, Comment. Math. Univ. Carolin. 26 (1985) 565–577.

[7] S.I. Pohožaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. 6 (1965) 1408–1411, English transl. in Dokl. Akad. Nauk SSSR 165 (1965) 33–36.

[8] P. Pucci, J. Serrin, A general variational identity, Indiana Univ. Math. J. 35 (1986) 681–703.

[9] W. Reichel, Supercritical variational problems with unique critical points, in: Proceedings of the 4th European Conference on Elliptic and Parabolic Problems, Rolduc, June 2001, World Scientific, 2002, pp. 253–262.

[10] W. Reichel, H. Zou, Non-existence results for semilinear elliptic systems via moving spheres, J. Differential Equations 161 (2000).

[11] R. Schaaf, Uniqueness for semilinear elliptic problems—supercritical growth and domain geometry, Adv. Differential Equations 5 (2000) 1201–1220.